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On the rate of pointwise divergence of Fourier and wavelet series in L^p

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Abstract

Let $g \in L^p(\mathbb{T})$, $1 < p < \infty$. We show that the set of points where the Fourier partial sums $S_n g(x)$ diverge as fast as n^β has Hausdorff dimension less or equal to $1 - \beta p$. A comparable result holds for wavelet series. Conversely, we show that this inequality is sharp and depends only on the Hausdorff dimension of the set of divergence.

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1. Introduction

After Du Bois Raymond's construction of a continuous function whose Fourier series diverges at one point [5], Haar introduced an orthonormal basis of L^2 in which the expansion of continuous functions converge uniformly on compact sets. In further developments of the theory of Fourier series, Kahane and Katznelson [12] proved that, given any F_σ set $A \subset \mathbb{T} := \mathbb{R}/\mathbb{Z}$ of Lebesgue measure zero, there exists a continuous function whose Fourier series diverges everywhere on A; meanwhile the Haar basis became the prototype of wavelet bases, and wavelet expansions of continuous functions also converge uniformly on compact sets [20]. Here we see one of the main differences between Fourier and wavelet bases. Another difference is that wavelets yield unconditional bases of L^p, $1 < p < \infty$; this is false for the Fourier basis if $p \neq 2$. In this paper,

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we wish to compare wavelet and Fourier bases on the matter of pointwise divergence for functions in L^p.

Since the works of Carleson [2] and Hunt [9], we know that the Fourier series of a function in $L^p(\mathbb{T})$, $1 < p \leqslant \infty$, converges almost everywhere. Our goal in this paper is to study the rate of divergence at the other points; we shall see that the set of points where the divergence is "fast" must be "small" in the sense of Hausdorff dimension. On the other hand, the wavelet expansion of a function in L^p also converges almost everywhere if $1 \leqslant p \leqslant \infty$ [14]. The rate of divergence turns out to have a limitation equivalent to the one found for Fourier series. In both cases, we demonstrate that our bounds are sharp by constructing a function whose Fourier or wavelet series pointwise diverging at a given rate on any set satisfying the dimension condition.

These results can be interpreted as a bound on the "multifractal spectrum of Fourier (or wavelet) divergence" of a function $f \in L^p$, in the same fashion as functions in Besov spaces have bounded multifractal spectrum of Hölder singularities [10,11].

Let us first introduce some notations.

1.1. Fourier transform

Let $\zeta \in \mathbb{R}$ and $e_\zeta : t \mapsto e^{2\pi i \zeta t}$. The continuous Fourier transform of $f \in L^1(\mathbb{R})$ is

$$\widehat{f} : \zeta \mapsto \int_{\mathbb{R}} f(x)\overline{e_\zeta}(x)\,dx.$$

A function $g \in L^1(\mathbb{T})$ is identified to a 1-periodic function on $\mathbb{R}$. Its Fourier transform is the tempered distribution

$$\widehat{g} : \zeta \mapsto \sum_{k\in\mathbb{Z}} \langle g, e_k\rangle\, \delta(\zeta - k),$$

where

$$\langle g, e_k\rangle := \int_{\mathbb{T}} g(t)\overline{e_k(t)}\,dt.$$

For $n \in \mathbb{N}$, we define the band-limiting operator S_n by $\widehat{S_n f} := \mathbf{1}_{[-n,n]}\widehat{f}$. If $g \in L^1(\mathbb{T})$, this corresponds to taking the partial sum of the Fourier series

$$S_n g : t \mapsto \sum_{k=-n}^{n} \langle g, e_k\rangle\, e_k(t).$$

We write $\mathcal{E}_n(\mathbb{R}) := S_n(L^1(\mathbb{R}))$ and $\mathcal{E}_n(\mathbb{T}) := S_n(L^1(\mathbb{T}))$, respectively, the sets of band-limited functions on $\mathbb{R}$ and $\mathbb{T}$. S_n is the orthogonal projection (with respect to the L^2 scalar product) on $\mathcal{E}_n$.

1.2. Wavelets

Let Φ and Ψ be the father and mother wavelets of a multiresolution analysis of $L^2(\mathbb{R})$: Ψ has at least one zero moment and the family of functions $\Phi_k : x \mapsto \phi(x-k)$ and $\Psi_{jk} : x \mapsto 2^{\frac{j}{2}}\psi(2^j x - k)$, $(j,k) \in \mathbb{N}\times\mathbb{Z}$, form an orthonormal basis of $L^2(\mathbb{R})$. We also require that Φ and Ψ are, for an $\varepsilon > 0$, piecewise ε-Lipschitz and rapidly decreasing functions.

Since we are interested in local properties and want to compare the wavelets with the Fourier series, we shall focus on periodic functions. An orthonormal basis of $L^2(\mathbb{T})$ is formed by the periodized wavelets [3,16]

$$\phi : x \mapsto \sum_{k\in\mathbb{Z}} \Phi_k(x) = 1$$

and for $j \in \mathbb{N}, 0 \leqslant k < 2^j$,

$$\psi_{jk} : x \mapsto \sum_{l\in\mathbb{Z}} \Psi_{jk}(x-l).$$

Similarly to the Fourier partial sums, we define the wavelet partial sums

$$T_{j_0} g : t \mapsto \langle g, 1\rangle + \sum_{j=0}^{j_0} \sum_{k=0}^{2^j-1} \langle g, \psi_{jk}\rangle \psi_{jk}(t),$$

as well as

$$T^*_{j_0} g : t \mapsto |\langle g, 1\rangle| + \sum_{j=0}^{j_0} \sum_{k=0}^{2^j-1} \left|\langle g, \psi_{jk}\rangle \psi_{jk}(t)\right|.$$

1.3. Hausdorff dimension

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function (dimension function), the ϕ-Hausdorff outer measure of a set $E \subset \mathbb{R}$ is

$$\mathcal{H}^\phi(E) := \lim_{\varepsilon\to 0} \inf_{r\in R_\varepsilon(E)} \sum_{B\in r} \phi(|B|),$$

$R_\varepsilon(E)$ being the set of countable coverings of E with intervals B of length $|B| \leqslant \varepsilon$. When $\phi_s(x) = x^s$, we write for short $\mathcal{H}^s$ instead of $\mathcal{H}^{\phi_s}$. The Hausdorff dimension of a set E is

$$\dim_H(E) := \sup\left\{s, \mathcal{H}^s(E) > 0\right\}.$$

This definition is not changed if the coverings are restricted to dyadic intervals [7]. Moreover, if E has Hausdorff measure zero, there exists not only a covering such that the above series converges, but an infinite covering as well:

Lemma 1. *If $\mathcal{H}^\phi(E) = 0$, then there exists a sequence E_j, union of N_j dyadic intervals of size 2^{-j}, such that $\sum_j N_j \phi(2^{-j}) \leqslant 2$ and $E \subset \limsup_j E_j$.*

Proof. Since $\mathcal{H}^\phi(E) = 0$, for all $j_0 \in \mathbb{N}$ there is a covering $\bigcup_{I\in r_{j_0}} I \supset E$, where r_{j_0} is a set of dyadic intervals of size $\leqslant 2^{-j_0}$, satisfying $\sum_{I\in r_{j_0}} \phi(I) \leqslant 2^{-j_0}$. Let N_j be the number of all the intervals of size 2^{-j} in $\bigcup_{j_0} r_{j_0}$, and E_j their union. Then $E \subset \limsup_j E_j$ and $\sum_j N_j \phi(2^{-j}) \leqslant \sum_{j_0} 2^{-j_0} = 2$. □

If B is an interval and $\lambda > 0$, λB will denote the interval with same center as B and length $\lambda|B|$.

2. Fourier series

Let us state our main theorem for Fourier series.

Theorem 2. *Let* $g \in L^p(\mathbb{T})$, $1 < p < \infty$. *For* $\beta \geqslant 0$, *let*

$$E(\beta) := \left\{ x, \limsup_{n\to\infty} n^{-\beta} |S_n g(x)| > 0 \right\}.$$

Then $\dim_H(E(\beta)) \leqslant 1 - \beta p$.

Conversely, given a set E *such that* $\dim_H(E) < 1 - \beta p$, *there exists a function* $g \in L^p(\mathbb{T})$ *such that for all* $x \in E$, $\limsup_n n^{-\beta}|S_n g(x)| = \infty$.

This theorem is a direct consequence of Corollaries 10 and 13, derived from the more precise Propositions 9 and 11 below.

2.1. Preliminary results

In the long and fertile story of Fourier series in L^p spaces, M. Riesz proved first the uniform continuity of the operators S_n:

Proposition 3. *If* $1 < p < \infty$, *there exists* $C_p < \infty$ *such that, for all* $n \in \mathbb{N}$,

$$\|S_n g\|_{L^p} \leqslant C_p \|g\|_{L^p},$$

if $g \in L^p(\mathbb{R}$ *or* $\mathbb{T})$.

Later, this was much improved [2,9] (see also [1]):

Theorem 4 (*Carleson, Hunt*). *Let* $S^* g(x) := \sup_n |S_n g(x)|$. *The operator* S^* *is a continuous endomorphism of* $L^p(\mathbb{R}$ *or* $\mathbb{T})$ *for all* $p \in (1, \infty)$.

We also recall the classical Nikolsky inequality (see [17]):

Proposition 5. *Let* $g \in \mathcal{E}_n(\mathbb{R}$ *or* $\mathbb{T})$. *If* $1 \leqslant p \leqslant q \leqslant \infty$, *then*

$$\|g\|_{L^q} \leqslant n^{\frac{1}{p}-\frac{1}{q}} \|g\|_{L^p}.$$

We shall need a localized version of this inequality (for $q = \infty$), where we multiply $g \in \mathcal{E}_n(\mathbb{T})$ by a compactly supported "window" function χ_n of width proportional to $\frac{1}{n}$. Ideally, we would want $\left\|\chi_n g\right\|_{L^\infty(\mathbb{R})} \leqslant C n^{\frac{1}{p}} \left\|\chi_n g\right\|_{L^p(\mathbb{R})}$, but this would require that $\widehat{\chi}$ also has compact support (see proof of Lemma 8). Since this cannot be, some loss in the inequality is necessary. At best, $\widehat{\chi}$ can have almost-exponential decrease.

Lemma 6. *Let* $\gamma > 0$ *and* $H(\gamma)$ *be the set of functions* f *for which there exist* C_f *and* $C'_f > 0$ *such that*, $\forall \zeta \in \mathbb{R}$, $\left|\widehat{f}(\zeta)\right| \leqslant C_f e^{-C'_f|\zeta|^\gamma}$. *Then* $H(\gamma)$ *contains a non-zero compactly supported function* χ *if and only if* $\gamma < 1$.

Note that $H(\gamma)$ is an algebra for pointwise multiplication, so χ can be taken non-negative. Also, up to a rescaling of χ, the constants C_χ and C'_χ be chosen arbitrarily in $(0, \infty)$.

Proof of Lemma 6. Let us first prove the "only if" part. If the function f satisfies the inequality

$$\forall \xi \in \mathbb{R}, \quad \left|\hat{f}(\xi)\right| \leqslant C_f e^{-C'_f |\xi|^\gamma},$$

with some $\gamma \geqslant 1$, then we also have

$$\forall \xi \in \mathbb{R}, \quad \left|\hat{f}(\xi)\right| \leqslant B_f e^{-B'_f |\xi|}.$$

This condition implies, by the Paley–Wiener theorem, that f has an analytical continuation to the domain $\{z : |\Im z| < B'_f\}$, therefore cannot be compactly supported if it is non-zero.

Conversely, suppose that $\gamma < 1$ and let χ be defined as in [18] by its Fourier transform.

$$\widehat{\chi}(\zeta) := \left(\frac{\sin(\zeta)}{\zeta}\right)^2 \prod_{k=1}^{\infty} \frac{\sin(\zeta/k^{\frac{1}{\gamma}})}{\zeta/k^{\frac{1}{\gamma}}}.$$

With the Paley–Wiener theorem, it can be seen that χ has compact support. Let $|\zeta| \geqslant 1, n := \left\lfloor |\zeta|^\gamma \right\rfloor$, and remark that

$$|\widehat{\chi}(\zeta)| \leqslant \prod_{k=1}^{n} \left|\frac{\sin(\zeta/k^{\frac{1}{\gamma}})}{\zeta/k^{\frac{1}{\gamma}}}\right| \leqslant \prod_{k=1}^{n} \frac{k^{\frac{1}{\gamma}}}{|\zeta|} \leqslant \frac{M_n(\gamma)}{n^{n/\gamma}} \sim_{n\to\infty} \left(\sqrt{2\pi n} e^{-n}\right)^{\frac{1}{\gamma}}.$$

So there exists C such that $|\widehat{\chi}(\zeta)| \leqslant C e^{-\frac{n+1}{2\gamma}} \leqslant C e^{-\frac{1}{2\gamma}|\zeta|^\gamma}$, hence $\chi \in H(\gamma)$. □

Remark 7. Using the Denjoy–Carleson theorem, one can actually show that, given a positive decreasing function r, the necessary and sufficient condition for the set of functions with Fourier decrease $\left|\widehat{f}(\zeta)\right| \leqslant r(|\zeta|)$ to possess a non-zero compactly supported function is that $\int_1^\infty \frac{\log(r(v))}{v^2}\, dv > -\infty$. This would slightly improve the subsequent results, but to the price of some unwanted complication.

In the following, $v > 1$ is fixed and χ will be a compactly supported function in $H(\frac{1}{v})$ such that, for all $y \in \mathbb{R}$, $0 \leqslant \chi(y) \leqslant \chi(0) = 1$ (see proof below). Given $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we define $\chi_{x,n} : y \mapsto \chi(n(y - x))$, which can be viewed as a localization window at scale n^{-1} around x.

Lemma 8. *For all $\theta > 0$, $1 < p < \infty$, there exist a constant C and a function χ as above such that, for all $n \in \mathbb{N}$, $g \in \mathcal{E}_n(\mathbb{T})$ and $x \in \mathbb{R}$,*

$$\left\|\chi_{x,n} g\right\|_{L^\infty(\mathbb{R})} \leqslant C\left(n^{\frac{1}{p}} \log(n)^{\frac{v}{p}} \left\|\chi_{x,n} g\right\|_{L^p(\mathbb{R})} + n^{-\theta} \|g\|_{L^p(\mathbb{T})}\right).$$

Proof. Lemma 6 provides a compactly supported, non-negative, continuous function in $H(\frac{1}{v})$; this vector space being invariant by translations, one can ask that this function χ reaches its maximum value $= 1$ at 0. Note that $\widehat{\chi_{x,n}}(\zeta) = \frac{1}{n}\widehat{\chi}\left(\frac{\zeta}{n}\right) e^{-2\pi i x \zeta}$. If $g \in \mathcal{E}_n(\mathbb{T})$ (a trigonometric polynomial), then $\widehat{g}(\zeta) = \sum_{j=-n}^{n} \langle g, e_j \rangle \delta(\zeta - j)$. We write

$$\widehat{\chi_{x,n} g}(\zeta) = \underbrace{\mathbf{1}_{|\zeta| \leqslant n \log(n)^v} \widehat{\chi_{x,n} g}(\zeta)}_{\widehat{f_1}(\zeta)} + \underbrace{\mathbf{1}_{|\zeta| > n \log(n)^v} \widehat{\chi_{x,n} g}(\zeta)}_{\widehat{f_2}(\zeta)}.$$

Since $\widehat{f_1}$ is supported in the set $\{|\zeta| \leqslant n \log(n)^v\}$, by Propositions 5 and 3,

$$\|f_1\|_{L^\infty} \leqslant n^{\frac{1}{p}} \log(n)^{\frac{v}{p}} \|f_1\|_{L^p} \leqslant C_p n^{\frac{1}{p}} \log(n)^{\frac{v}{p}} \left\|\chi_{x,n} g\right\|_{L^p}.$$

On the other hand,

$$\begin{aligned}\widehat{f_2}(\zeta) &= \mathbf{1}_{|\zeta| > n \log(n)^v} (\widehat{\chi_{x,n}} \star \widehat{g})(\zeta) \\ &= \sum_{j=-n}^{n} \frac{\langle g, e_j \rangle}{n} \mathbf{1}_{|\zeta| > n \log(n)^v} \widehat{\chi}\left(\frac{\zeta - j}{n}\right) e^{-2\pi i x(\zeta - j)}.\end{aligned}$$

Let us bound the L^1 norm of the terms of this sum: for all $-n \leqslant j \leqslant n$,

$$\begin{aligned}\int_{n \log(n)^v}^{\infty} \frac{1}{n} \left|\widehat{\chi}\left(\frac{\zeta - j}{n}\right)\right| d\zeta &= \int_{\log(n)^v - \frac{j}{n}}^{\infty} |\widehat{\chi}(u)|\, du \\ &\leqslant \int_{\frac{1}{2}\log(n)^v}^{\infty} C_\chi e^{-C'_\chi u^{\frac{1}{v}}}\, du \\ &\leqslant C_\chi \int_{\frac{1}{2}\log(n)^v}^{\infty} \frac{C'_\chi}{2v} u^{\frac{1}{v} - 1} e^{-\frac{C'_\chi}{2} u^{\frac{1}{v}}}\, du \\ &\leqslant C_\chi e^{-\frac{C'_\chi}{4}\log(n)} \\ &\leqslant C_\chi n^{-(\theta+1)},\end{aligned}$$

if n is large enough so that $u \geqslant \frac{1}{2}\log(n)^v \Rightarrow e^{-\frac{C'_\chi}{2} u^{\frac{1}{v}}} \leqslant \frac{C'_\chi}{2v} u^{\frac{1}{v} - 1}$ and if we chose χ such that $C'_\chi \geqslant 4(\theta + 1)$. The other half of the integral has the same bound, so finally, by the inverse Fourier transform,

$$\begin{aligned}\|f_2\|_{L^\infty} \leqslant \|\widehat{f_2}\|_{L^1} &\leqslant \sum_{j=-n}^{n} \int_{|\zeta| > n \log(n)^v} \left| \frac{\langle g, e_j \rangle}{n} \widehat{\chi}\left(\frac{\zeta - j}{n}\right) \right| d\zeta \\ &\leqslant 2C_\chi n^{-(\theta+1)} \sum_{j=-n}^{n} |\langle g, e_j \rangle| \\ &\leqslant 6C_\chi n^{-\theta} \|g\|_{L^p(\mathbb{T})},\end{aligned}$$

using Parseval's theorem, Schwarz's inequality, and Proposition 3. $\square$

Note that the choice of χ actually depends on θ: the larger θ, the larger the support of χ.

2.2. *Upper bound on the Hausdorff dimension*

Proposition 9. *Let* $g \in L^p(\mathbb{T})$, $1 < p < \infty$. *Let* $\tau : [0, \infty) \to (0, \infty)$ *be an increasing function and* $E(\tau) := \left\{x, \limsup_n \frac{|S_n g(x)|}{\tau(n)} = \infty\right\}$. *If* $v > 1$ *and* $\phi(s) = O_{s \to 0^+}\left(\frac{s\tau(s^{-1})^p}{\log(s^{-1})^v}\right)$, *then* $\mathcal{H}^\phi(E(\tau)) = 0$.

Proof. Let $M \in \mathbb{N}$ and $E_M := \left\{x, \limsup_n \frac{|S_n g(x)|}{\tau(n)} > M\right\}$. Let $\theta > 0$ and $\chi \in H(\frac{1}{v})$ be the function from Lemma 8, supported in $[-K, K]$. We also fix an $\eta > 0$. For every $x \in E_M$, there exist infinitely many n_x such that $\left|S_{n_x} g(x)\right| \geqslant M\tau(n_x)$. We chose n_x large enough so that $n_x^{-1} < \frac{\eta}{10K}$ and $n_x^{-\theta} \sup_n \|S_n g\|_{L^p(\mathbb{T})} \leqslant \frac{M}{2C}\tau(n_x)$ (the constant C from Lemma 8, depending only on θ and p). Then, defining $B_x := [x - \frac{K}{n_x}, x + \frac{K}{n_x}]$, we have

$$
\begin{aligned}
\left\|S_{n_x} g\right\|_{L^p(B_x)} &\geqslant \left\|\chi_{x,n_x} S_{n_x} g\right\|_{L^p(\mathbb{R})} \\
&\geqslant \frac{\frac{1}{C}\left\|\chi_{x,n_x} S_{n_x} g\right\|_{L^\infty(\mathbb{R})} - n_x^{-\theta}\left\|S_{n_x} g\right\|_{L^p(\mathbb{T})}}{n_x^{\frac{1}{p}} \log(n_x)^{\frac{v}{p}}} \\
&\geqslant \frac{\frac{1}{C}\left|S_{n_x} g(x)\right| - \frac{M}{2C}\tau(n_x)}{n_x^{\frac{1}{p}} \log(n_x)^{\frac{v}{p}}} \\
&\geqslant \frac{M}{2C} \frac{\tau(n_x)}{n_x^{\frac{1}{p}} \log(n_x)^{\frac{v}{p}}}.
\end{aligned}
$$

By the "5r-covering theorem" [8,15], we can extract from $\{B_x, x \in E_M\}$ a countable family of disjoint intervals $B_i, i \in \mathbb{N}$, of size $\frac{2K}{n_i}$, such that $E_M \subset \bigcup_{i \in \mathbb{N}} 5B_i$. Then,

$$
\begin{aligned}
\int_{\mathbb{T}} \left|S^* g(x)\right|^p dx &\geqslant \sum_{i \in \mathbb{N}} \int_{B_i} \left|S^* g(x)\right|^p dx \\
&\geqslant \sum_{i \in \mathbb{N}} \int_{B_i} \left|S_{n_i} g(x)\right|^p dx \\
&\geqslant \sum_{i \in \mathbb{N}} \left(\frac{M}{2C}\right)^p \frac{\tau(n_i)^p}{n_i \log(n_i)^v}.
\end{aligned}
$$

We thus have found that the family $\{5B_i, i \in \mathbb{N}\}$ is an η-covering of E_M satisfying

$$
\begin{aligned}
\sum_{i \in \mathbb{N}} \phi(|5B_i|) &\leqslant C_1 \sum_{i \in \mathbb{N}} \frac{\frac{10K}{n_i}\tau(\frac{n_i}{10K})^p}{\log\left(\frac{n_i}{10K}\right)^v} \\
&\leqslant C_2 M^{-p} \left\|S^* g\right\|_{L^p}^p,
\end{aligned}
$$

which is finite by the Carleson–Hunt theorem. Note that this bound is independent of η, and thus also bounds $\mathcal{H}^\phi(E_M)$. But $E(\tau) = \bigcap_M E_M$, hence $\mathcal{H}^\phi(E(\tau)) = \lim_{M\to\infty} \mathcal{H}^\phi(E_M) = 0$. □

We should point out the fact that, although it makes the proof more elegant, the full strength of the Carleson–Hunt theorem is not used here. Indeed, when $\tau(n) = n^0$, Proposition 9 yields only $\mathcal{H}^\phi(E(\tau)) = 0$ for $\phi(x) = \frac{x}{\log(x^{-1})^v}$, whereas the simple fact that $S^* g \in L^p$ implies that $\mathcal{H}^1(E(\tau)) = 0$ (the Fourier series is bounded Lebesgue-almost everywhere). Similarly, one might expect that for $\tau(n) = n^\beta$, $\mathcal{H}^{1-\beta p}(E(\tau)) = 0$. This shortcoming seems to be inherent to the localization technique employed; we can only conjecture that v could be replaced by 0 in the proposition.

Corollary 10. *Let $g \in L^p(\mathbb{T})$, $1 < p < \infty$. For $\beta \geqslant 0$, let*

$$E(\beta) := \left\{ x, \limsup_{n\to\infty} n^{-\beta} |S_n g(x)| > 0 \right\}.$$

Then $\dim_H(E(\beta)) \leqslant 1 - \beta p$.

Naturally, a negative Hausdorff dimension means that the set is empty.

Proof of Corollary 10. Take $\tau(n) := \frac{n^\beta}{\log(n)}$ and $\phi(x) := x^{1-\beta p+\varepsilon}$ for an arbitrary $\varepsilon > 0$. Then $E(\beta) \subset E(\tau)$, and applying Proposition 9 yields $\mathcal{H}^{1-\beta p+\varepsilon}(E(\tau)) = 0$. It follows that $\dim_H(E(\beta)) \leqslant \dim_H(E(\tau)) \leqslant 1 - \beta p + \varepsilon$. □

2.3. Optimality of the upper bound

Our converse to Proposition 9 is the construction of a function, showing that the above bound is optimal and depends only on the Hausdorff dimension of the set.

Proposition 11. *Let $1 < p < \infty$, $\tau : [0, \infty) \to (0, \infty)$ be an increasing function and ϕ be a dimension function such that*

$$\int_0^1 \left(\frac{s\tau(s^{-1})^p}{\phi(s)} \right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty. \tag{1}$$

For every set $E \subset \mathbb{T}$ satisfying $\mathcal{H}^\phi(E) = 0$, there exists $g \in L^p(\mathbb{T})$ such that for all $x \in E$, $\limsup_n \frac{|S_n g(x)|}{\tau(n)} = \infty$.

Proof. Let us first remark that whenever (1) is true, one can find a non-decreasing $\tilde{\tau}$ such that $\lim_{n\to\infty} \frac{\tilde{\tau}(n)}{\tau(n)} = \infty$ but still

$$\int_0^1 \left(\frac{s\tilde{\tau}(s^{-1})^p}{\phi(s)} \right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty. \tag{2}$$

By Lemma 1, we have an infinite covering of E by unions E_j of N_j dyadic intervals of length 2^{-j}. Let $\chi_j : x \mapsto 1 - \min(1, 2^j d(x, E_j))$: this function satisfies $\chi_j(x) = 1$ if $x \in E_j$, $\chi_j(x) = 0$ outside of a set of measure $\leqslant 3N_j 2^{-j}$, and $\left\| \chi_j' \right\|_{L^\infty} \leqslant 2^j$. Introducing the Fejér sum

$$\sigma_{2^j} g : t \mapsto \frac{1}{2^j} \sum_{k=0}^{2^j-1} S_k g(t)$$

and the Fejér kernel $F_{2^j} : x \mapsto \frac{1}{2^j} \left(\frac{\sin(2^j \pi x)}{\sin(\pi x)} \right)^2$, we have

$$\sigma_{2^j} \chi_j(x) - \chi_j(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\chi_j(x+y) - \chi_j(x)) F_{2^j}(y)\, dy$$

(see e.g. [6]). We claim that (if $j \geqslant 3$) this integral is absolutely bounded by $\frac{1}{2}$ (see Lemma 12 below). It follows that for all $x \in E_j$, $\sigma_{2^j}\chi_j(x) \geqslant \frac{1}{2}$. On the other hand,

$$\left\|\sigma_{2^j}\chi_j\right\|_{L^p} \leqslant \left\|F_{2^j}\right\|_{L^1}\left\|\chi_j\right\|_{L^p} = \left\|\chi_j\right\|_{L^p} \leqslant (3N_j 2^{-j})^{\frac{1}{p}}.$$

Let us now define

$$g := \sum_{j\in\mathbb{N}} \tilde{\tau}(2^j) e_{3.2^j}\sigma_{2^j}\chi_j.$$

The frequency shift (multiplication by $e_{3.2^j}$) is here to prevent “interferences”: since $\sigma_{2^j}\chi_j$ has Fourier support in $\{1-2^j, \dots, 2^j-1\}$, the Fourier support of $e_{3.2^j}\sigma_{2^j}\chi_j$ is in $\{2^{j+1}+1, 2^{j+2}-1\}$, disjoint from the other terms. Therefore if $l \in \mathbb{N}$,

$$S_{2^{l+2}}g(x) = S_{2^{l+1}}g(x) + \tilde{\tau}(2^l)e_{3.2^l}(x)\sigma_{2^l}\chi_l(x),$$

so that

$$\max\left(\left|S_{2^{l+1}}g(x)\right|, \left|S_{2^{l+2}}g(x)\right|\right) \geqslant \tfrac{1}{2}\tilde{\tau}(2^l)\sigma_{2^l}\chi_l(x).$$

If $x \in E_l$, then there exists n (either 2^{l+1} or 2^{l+2}), such that $|S_n g(x)| \geqslant \frac{1}{4}\tilde{\tau}(n)$. But for an $x \in E$, this happens for infinitely many l, so $\limsup_n \frac{|S_n g(x)|}{\tau(n)} = \infty$.

Furthermore, we have the following estimate on the L^p-norm of g:

$$\begin{aligned} 3^{-\frac{1}{p}}\|g\|_{L^p} &\leqslant \sum_{j\in\mathbb{N}} \tilde{\tau}(2^j)(N_j 2^{-j})^{\frac{1}{p}} \\ &\leqslant \sum_{j\in\mathbb{N}} (N_j\phi(2^{-j}))^{\frac{1}{p}} \frac{\tilde{\tau}(2^j)2^{-\frac{j}{p}}}{\phi(2^{-j})^{\frac{1}{p}}} \\ &\leqslant \left(\sum_{j\in\mathbb{N}} N_j\phi(2^{-j})\right)^{\frac{1}{p}} \left(\sum_{j\in\mathbb{N}} \left(\frac{\tilde{\tau}(2^j)^p 2^{-j}}{\phi(2^{-j})}\right)^{\frac{1}{p-1}}\right)^{1-\frac{1}{p}}, \end{aligned}$$

the first term being finite by Lemma 1 and the second one by (2). □

Lemma 12. *For any $j \geqslant 3$, if θ is a function such that $\|\theta\|_{L^\infty} \leqslant 1$ and $\left\|\theta'\right\|_{L^\infty} \leqslant 2^j$, then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\theta(y)| F_{2^j}(y)\,dy \leqslant 1/2.$$

Proof. We split the integral in two parts, $|y| < 2^{-j}$ and on $2^{-j} < |y| < \frac{1}{2}$. The first part is bounded by

$$u_j := 2\int_0^{2^{-j}} \left(\frac{\sin(2^j\pi y)}{\sin(\pi y)}\right)^2 y\,dy.$$

Doing a change of variables and using the fact that $\sin(\frac{\pi y}{2}) \geqslant \frac{\sin(\pi y)}{2}$ we see that u_j is decreasing; numerically $u_3 = 0.2496183586\ldots$ so for all $j \geqslant 3$, $u_j < \frac{1}{4}$.

For the second part note that if $j \geqslant 3$, then $\frac{1}{2} > y > 2^{-j}$ implies that $\sin(\pi y) > 2^{1-j}$ so actually $F_{2^j}(y) \leqslant \frac{\sin(2^j \pi y)^2}{2}$ and

$$\int_{2^{-j}<|y|<\frac{1}{2}} |\theta(y)| F_{2^j}(y)\, dy \leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin(2^j \pi y)^2}{2}\, dy = \frac{1}{4}. \qquad \square$$

Corollary 13. *If* $1 < p < \infty$ *and* E *is a set such that* $\dim_H(E) < 1 - \beta p$, *then there exists* $g \in L^p(\mathbb{T})$ *such that for all* $x \in E$, $\limsup_n n^{-\beta} |S_n g(x)| = \infty$.

Proof. Take $\tau(n) := n^\beta$ and $\phi(x) := x^\alpha$, where $\dim_H(E) < \alpha < 1 - \beta p$. $\square$

3. Wavelet series

If g is a function in $L^p(\mathbb{T})$, its wavelet coefficients (with L^2 normalization) are trivially bounded by the Hölder inequality: $|\langle g, \psi_{jk}\rangle| \leqslant C_\psi \|g\|_{L^p} 2^{(\frac{1}{p}-\frac{1}{2})j}$, where C_ψ depends only on ψ; so at any point $x \in \mathbb{T}$, $T_j g(x)$ (and even $T_j^* g(x)$) cannot diverge faster than $2^{\frac{j}{p}}$; on the other hand, this series converges almost everywhere [14]. As in the case of Fourier series, we have the following finer result on pointwise divergence rates.

Theorem 14. *Let* $g \in L^p(\mathbb{T})$, $1 < p < \infty$. *For* $\beta \geqslant 0$, *let*

$$E(\beta) := \left\{ x, \limsup_{j\to\infty} 2^{-\beta j} T_j^* g(x) > 0 \right\}.$$

Then $\dim_H(E(\beta)) \leqslant 1 - \beta p$.

Conversely, if ψ *is the Haar wavelet, given a set* E *such that* $\dim_H(E) < 1 - \beta p$, *there exists* $g \in L^p(\mathbb{T})$ *such that for all* $x \in E$,

$$\limsup_{j\to\infty} 2^{-\beta j} |T_j g(x)| = \infty.$$

This theorem a direct consequence of Propositions 18 and 19 below. Note that the series of absolute values $T_j^* g$ has essentially the same bound on its multifractal spectrum of pointwise divergence than $T_j g$; this is in contrast with the Fourier series.

In the case where $p \geqslant 2$ and the wavelet is compactly supported, we get a more precise result on the Hausdorff measures of the divergence sets (Proposition 16).

3.1. Upper bound on the Hausdorff dimension

As mentioned before, wavelets provide an unconditional basis of L^p. More precisely, the Calderon–Zygmund theory can be restated in terms of periodized wavelets under rather weak regularity conditions. According to Corollary 4.5 of [4], if as specified in 1.2 ψ is piecewise ε-Lipschitz, then we know that $g \in L^p(\mathbb{T})$ if and only if the function

$$\tilde{g} : x \mapsto \left(\sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} |\langle g, \psi_{jk}\rangle \psi_{jk}(x)|^2 \right)^{\frac{1}{2}}$$

is also in L^p. Crudely speaking, this means that we cannot have too many “big” wavelet coefficients; we shall translate this in terms of Hausdorff dimension.

Let us start with a simpler case.

Lemma 15. *Let $g \in L^p(\mathbb{T})$, $2 \leqslant p < \infty$, $m > 0$ and $l \in \mathbb{N}$ be fixed. Let $\tau : [0, \infty) \to (0, \infty)$ be an increasing function and define*

$$K_j := \left\{ k, |\langle g, \psi_{jk} \rangle| \geqslant \frac{2^{-\frac{j}{2}} \tau(2^j)}{m} \right\},$$

$$I_{jk} := [(k-l)2^{-j}, (k+l)2^{-j}]$$

and

$$F^m(\tau) := \limsup_{j \to \infty} \bigcup_{k \in K_j} I_{jk}.$$

Then $\mathcal{H}^\phi(F^m(\tau)) = 0$, with $\phi(s) := s\tau(s^{-1})^p$.

Proof. Let $n_j := \#(K_j)$. Without loss of generality, we can assume that there exists $0 < a < 1$ such that $|\Psi(x)| \geqslant a$ when $x \in [0, a]$. Let $\chi_{jk} := \mathbf{1}_{[k2^{-j}, (k+a)2^{-j}]}$. By our assumption above, $2^j \chi_{jk}(x) \leqslant a^{-2} |\psi_{jk}(x)|^2$ for all x. Note that

$$n_j = \frac{2^j}{a} \int_{\mathbb{T}} \left(\sum_{k \in K_j} \chi_{jk}(x) \right)^{\frac{p}{2}} dx$$

and thus

$$\begin{aligned} a \sum_{j=0}^{\infty} n_j \phi(2^{-j}) &= \sum_{j=0}^{\infty} 2^{p(\frac{1}{p}-\frac{1}{2})j} \phi(2^{-j}) \int_{\mathbb{T}} \left(\sum_{k \in K_j} 2^j \chi_{jk}(x) \right)^{\frac{p}{2}} dx \\ &\leqslant \int_{\mathbb{T}} \left(\sum_{j=0}^{\infty} \sum_{k \in K_j} \left(2^{(\frac{1}{p}-\frac{1}{2})j} \phi(2^{-j})^{\frac{1}{p}} \right)^2 2^j \chi_{jk}(x) \right)^{\frac{p}{2}} dx \\ &\leqslant \int_{\mathbb{T}} \left(\sum_{j=0}^{\infty} \sum_{k \in K_j} \left(2^{-\frac{j}{2}} \tau(2^j) \right)^2 a^{-2} |\psi_{jk}(x)|^2 \right)^{\frac{p}{2}} dx \\ &\leqslant \int_{\mathbb{T}} \left(\sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} m a^{-2} |\langle g, \psi_{jk} \rangle \psi_{jk}(x)|^2 \right)^{\frac{p}{2}} dx, \end{aligned}$$

which is finite because $\tilde{g} \in L^p$. But for all $j_0 \in \mathbb{N}$, $F^m(\tau)$ is naturally covered by the I_{jk}, $j \geqslant j_0$, $k \in K_j$; the above convergence implies that

$$\lim_{j_0 \to \infty} \sum_{j=j_0}^{\infty} \sum_{k \in K_j} \phi(|I_{jk}|) = 0,$$

therefore the ϕ-Hausdorff measure is zero. □

Proposition 16. *Let $g \in L^p(\mathbb{T})$, $2 \leqslant p < \infty$, and assume that the mother wavelet Ψ has compact support. Let $\beta > 0$, and*

$$E(\beta) := \left\{ x, \limsup_{j_0 \to \infty} 2^{-\beta j_0} T^*_{j_0} g(x) > 0 \right\}.$$

If $\beta > \frac{1}{p}$, then $E(\beta) = \emptyset$; else $\mathcal{H}^{1-\beta p}(E(\beta)) = 0$.

Proof. Note that in the case of compactly supported Ψ, if j is large enough, then $\psi_{jk} = \Psi_{jk}$, so we can forget that we are working with periodized wavelets.

Considering the remark that precedes Theorem 14, we can assume that $0 < \beta \leqslant \frac{1}{p}$. Let $m > 0$ and l be larger than the support of Ψ. Let

$$E^m(\beta) := \left\{ x, \limsup_{j_0 \to \infty} 2^{-\beta j_0} T^*_{j_0} g(x) > \frac{1}{m} \right\}.$$

We want to show that if $\tau(n) = n^\beta$, then for some constant C depending only on β, p and Ψ, we have $E^{\frac{m}{C}}(\beta) \subset F^m(\tau)$.

Suppose that $x \notin F^m(\tau)$. This means that, with at most a finite number of exceptions, $x \notin \bigcup_{k \in K_j} I_{jk}$. Since $l2^{-j}$ is larger than the support of ψ_{jk}, if $\psi_{jk}(x) \neq 0$, then $k \notin K_j$, and $|\langle g, \psi_{jk} \rangle| \leqslant \frac{1}{m} 2^{\left(\beta - \frac{1}{2}\right) j}$. So

$$\sum_{k=0}^{2^j - 1} |\langle g, \psi_{jk} \rangle \psi_{jk}(x)| = \sum_{k \notin K_j} |\langle g, \psi_{jk} \rangle \psi_{jk}(x)| \leqslant \frac{C_1}{m} 2^{\beta j},$$

where $C_1 := \sup_x \sum_k |\Psi_{0k}(x)| < \infty$. Since $\beta > 0$,

$$\sum_{j=0}^{j_0} \sum_{k=0}^{2^j - 1} |\langle g, \psi_{jk} \rangle \psi_{jk}(x)| \leqslant \frac{C}{m} 2^{\beta j_0},$$

with $C := C_1 \frac{2^\beta}{2^\beta - 1}$. So $x \notin E^{\frac{m}{C}}(\beta)$.

Using Lemma 15, we have proved that $\mathcal{H}^{1-\beta p}(E^{\frac{m}{C}}(\beta)) = 0$. To conclude, remark that $E(\beta) = \bigcup_{m \in \mathbb{N}} E^{\frac{m}{C}}(\beta)$, so by σ-additivity $\mathcal{H}^{1-\beta p}(E(\beta)) = 0$. □

If $p < 2$ or if the wavelet is not compactly supported, we need to adapt the proof, to the cost of some loss in precision. We are able to conclude only on the Hausdorff dimension.

Lemma 17. *Let* $g \in L^p(\mathbb{T})$, $1 < p < \infty$, $m > 0$ *and* $l \in \mathbb{N}$. *For* $\alpha \geqslant 0$ *and* $0 < \varepsilon < 1$, *we define*

$$K_j := \left\{ k, \left|\langle g, \psi_{jk}\rangle\right| \geqslant \frac{1}{m} 2^{-(\frac{\alpha-1}{p}+\frac{1}{2})j} \right\},$$

$$I^{\varepsilon}_{jk} := [(k - 2^{\varepsilon j})2^{-j}, (k + 2^{\varepsilon j})2^{-j}]$$

and

$$F^{m,\varepsilon}(\alpha) := \limsup_j \bigcup_{k \in K_j} I^{\varepsilon}_{jk}.$$

Then $\dim_H(F^{m,\varepsilon}(\alpha)) \leqslant \frac{\alpha}{1-\varepsilon}$.

Proof. We start as in the proof of Lemma 15, multiplying the terms of the sum by $j^{-\theta}$, with $\theta := \max(0, 2 - p)$:

$$\sum_{j=1}^{\infty} n_j 2^{-\alpha j} j^{-\theta} = \sum_{j=1}^{\infty} j^{-\theta} 2^{-2(\frac{\alpha-1}{p}+\frac{1}{2})j\frac{p}{2}} \int_{\mathbb{T}} \left(\sum_{k \in K_j} 2^j \chi_{jk}(x) \right)^{\frac{p}{2}} dx$$

using the Hölder inequality if $p < 2$,

$$\leqslant \int_{\mathbb{T}} C_0 \left(\sum_{j=1}^{\infty} \sum_{k \in K_j} 2^{-2(\frac{\alpha-1}{p}+\frac{1}{2})j} 2^j \chi_{jk}(x) \right)^{\frac{p}{2}} dx$$
$$< \infty$$

(here $C_0 = \left(\sum_{j=1}^{\infty} j^{-\theta\frac{2}{2-p}}\right)^{\frac{2-p}{p}}$ if $p < 2$, $C_0 = 1$ else). So $\sum_{j=1}^{\infty} n_j 2^{\alpha'(\varepsilon-1)j}$ is finite as soon as $\alpha' > \frac{\alpha}{1-\varepsilon}$; we conclude that $\dim_H(F^{m,\varepsilon})(\alpha) \leqslant \frac{\alpha}{1-\varepsilon}$. $\square$

Applying this lemma as in Proposition 11 will do the direct part of Theorem 14.

Proposition 18. *Let* $g \in L^p(\mathbb{T})$, $1 < p < \infty$. *Let* $\beta \geqslant 0$, *and*

$$E(\beta) := \left\{ x, \limsup_{j_0 \to \infty} 2^{-\beta j_0} \sum_{j=0}^{j_0} \sum_{k=0}^{2^j-1} \left|\langle g, \psi_{jk}\rangle \psi_{jk}(x)\right| > 0 \right\}.$$

If $\beta > \frac{1}{p}$, *then* $E(\beta) = \emptyset$; *else* $\dim_H(E(\beta)) \leqslant 1 - \beta p$.

Proof. Let $0 < \beta \leqslant \frac{1}{p}$, $m > 0$, and

$$E^m(\beta) := \left\{ x, \limsup_{j_0 \to +\infty} 2^{-\beta j_0} \sum_{j=0}^{j_0} \sum_{k=0}^{2^j-1} \left|\langle g, \psi_{jk}\rangle \psi_{jk}(x)\right| > \frac{1}{m} \right\}.$$

We chose an $\varepsilon > 0$ and we will show that if $\alpha := 1 - \beta p$, then for some constant C depending only on α, p, ε, and Ψ, we have $E^{\frac{m}{C}}(\beta) \subset F^{m,\varepsilon}(\alpha)$.

Suppose that $x \notin F^{m,\varepsilon}(\alpha)$. This means that, with at most a finite number of exceptions, $x \notin \bigcup_{k\in K_j} I^{\varepsilon}_{jk}$. So if $k \in K_j$, $\left|2^j x - k\right| \geqslant 2^{\varepsilon j}$, and Ψ being rapidly decreasing, for all θ there exists $C_\theta < +\infty$ such that $\left|\psi_{jk}(x)\right| \leqslant C_\theta 2^{-\theta\varepsilon j}$. It follows that $\sum_{k\in K_j} \left|\psi_{jk}(x)\right| \leqslant C_\theta 2^{(1-\varepsilon\theta)j}$. Moreover, since $g \in L^p(\mathbb{T})$, $\left|\langle g, \psi_{jk}\rangle\right| \leqslant C_\psi \|g\|_{L^p} 2^{(\frac{1}{p}-\frac{1}{2})j}$. We chose θ so that $\frac{1}{p} - \frac{1}{2} + 1 - \theta\varepsilon < -\frac{\alpha-1}{p}$, and then, if j is large enough,

$$\sum_{k\in K_j} \left|\langle g, \psi_{jk}\rangle \psi_{jk}(x)\right| \leqslant \frac{1}{m} 2^{-\frac{\alpha-1}{p} j}.$$

If $k \notin K_j$, we obtain as in the proof of Proposition 16

$$\sum_{k\notin K_j} \left|\langle g, \psi_{jk}\rangle \psi_{jk}(x)\right| \leqslant \frac{C_1}{m} 2^{-\frac{\alpha-1}{p} j}.$$

Adding up everything, and using the fact that $\alpha < 1$, we obtain for an adequate constant C:

$$\sum_{j=0}^{j_0} \sum_{k=0}^{2^j-1} \left|\langle g, \psi_{jk}\rangle \psi_{jk}(x)\right| \leqslant \frac{C}{m} 2^{\beta j_0}.$$

So $x \notin E^{\frac{m}{C}}(\beta)$. By Lemma 17, we deduce that $\dim_H(E^{\frac{m}{C}}(\beta)) \leqslant \frac{\alpha}{1-\varepsilon}$, and since $E(\beta) = \bigcup_{m\in\mathbb{N}} E^{\frac{m}{C}}(\beta)$, by σ-stability of the dimension, $\dim_H(E(\beta)) \leqslant \frac{\alpha}{1-\varepsilon}$. Finally we let $\varepsilon \to 0$. $\square$

3.2. Optimality of the upper bound

We now turn to the converse part, showing that the bound in Proposition 16 is optimal. To simplify, we assume from now on that ψ is the Haar wavelet.

Proposition 19. *Let $\alpha > 0$ and $1 \leqslant p < \infty$, and suppose that $\dim_H(E) < \alpha$. There exists $g \in L^p(\mathbb{T})$ such that for all $x \in E$, $\limsup_j 2^{\frac{\alpha-1}{p} j} T_j g(x) = \infty$.*

Proof. Let $\dim_H(E) < \alpha' < \alpha$ and define E_j and N_j as in Lemma 1, with $\phi(x) = x^{\alpha'}$. Let

$$g := \sum_{j=0}^{\infty} 2^{\frac{1-\alpha}{p} j} \mathbf{1}_{E_j}.$$

Then $\|g\|^p_{L^p} \leqslant \sum_{j=0}^{\infty} N_j 2^{-j} \left(2^{\frac{1-\alpha'}{p} j}\right)^p < \infty$. Moreover,

$$T_{j_0} g(x) = \sum_{j=0}^{j_0} 2^{\frac{1-\alpha'}{p} j} \mathbf{1}_{E_j}(x) + T_{j_0}\left(\sum_{j=j_0+1}^{\infty} 2^{\frac{1-\alpha'}{p} j} \mathbf{1}_{E_j}\right)(x) \tag{3}$$

$$\geqslant \sum_{j=0}^{j_0} 2^{\frac{1-\alpha'}{p} j} \mathbf{1}_{E_j}(x) \tag{4}$$

and if $x \in E$, $x \in E_{j_0}$ for infinitely many j_0, for which $T_{j_0} g(x) \geqslant 2^{\frac{1-\alpha'}{p} j_0}$. $\square$

Here we used two special properties of the Haar basis: in (3) the fact that indicatrix functions of dyadic intervals of size $\geqslant 2^{-j_0}$ are in the invariant subspace of T_{j_0}, in (4) the fact that the projection kernel is positive. This is in general false for other wavelets (causing the so-called Gibbs' phenomenon for wavelets [13,19]).

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